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Algebraic Bethe ansatz for the non-linear Schrödinger model: I. Multicomponent fields

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Abstract. The generalisation of the quantum inverse scattering method is used to derive the Bethe ansatz equations for a multicomponent non-linear Schrödinger model of a system with n species (or colours) of bosons or fermions. The matrix periodic boundary conditions for bosons and fermions are obtained and the relationship between them is discussed. The eigenstates for the infinite conserved quantities of the system are constructed. Lastly, we generalise these results to the non-linear Schrödinger model with the mirror-image delta potentials proposed by Schulz.

1. Introduction

A many-body system with two-body delta potentials has been extensively studied. As a field theory this system is the quantum non-linear Schrödinger model (QNSM), and the Hamiltonian is equal to

$$H=\int_{-L}^{L} \mathrm{d}x \left(\frac{\mathrm{d}q^{\dagger}}{\mathrm{d}x}\frac{\mathrm{d}q}{\mathrm{d}x}+c{:}q^{\dagger}qq^{\dagger}q{:}\right) \tag{1.1}$$

where $c>0$ is the coupling constant. We assume that the system consists of n species (or colours) of bosons or fermions and q is a $n\times 1$ matrix with the equal-time commutation relations

$$\begin{aligned} &q(x)_a q^{\dagger}(y)_b - pq^{\dagger}(y)_b q(x)_a = \delta_{a,b}\delta(x-y) \\ &q(x)_a q(y)_b - pq(y)_b q(x)_a = 0 \end{aligned} \tag{1.2}$$

where the factor $p=1$ and -1 for boson and fermion fields respectively. Here q is said to be a supermatrix with the structure $(m, n)\times(k, l)$, if m rows and k columns have even parity and the other n rows and l columns have odd parity. We indicate this as $q\sim(m, n)\times(k, l)$. Hence the parity of the matrix element q_{ij} may be written as $p(q_{ij})=p(i)+p(j)$, and $p(q_{ij})=0$ or 2 for a boson field and $p(q_{ij})=1$ for a fermion field.

The model (1.1) is integrable and some exact results were obtained. By using the Bethe ansatz (BA) technique the BA equations of the model were obtained for boson field $n=1$ by Lieb and Liniger (1963), for $\frac{1}{2}$-spin (or two species) fermion field $n=2$ by Yang (1967) and extended by Sutherland (1968) for the more-colour (or high-spin) system $n\geqslant 2$. By using the quantum inverse scattering method (QISM) the BA equations were derived for boson system $n=2$ by Kulish (1981), for the $\frac{1}{2}$-spin and more-species

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fermion system by Zhou and Zhao (1986) and extended by Fan *et al* (1986) for q having the structure $(m, n)\times(k, 0)$ (Zhou and Zhao 1987). The algebriac BA for the bosons $n=1$ can be found in papers by Faddeev (1981) and Thacker (1981). For a given q field we could generally take it as a supermatrix in two forms, i.e. $(m, n)\times(k, 0)$ or $(n, m)\times(0, k)$. We consider that the same BA equations should be obtained for $q\sim(m, n)\times(k, 0)$ and $(n, m)\times(0, k)$.

In this paper, we take q only having the structure $(n, 0)\times(0, 1)$ and $(0, n)\times(0, 1)$, or $q\sim(0, n)\times(1, 0)$ and $(n, 0)\times(1, 0)$ respectively for fermion and boson systems. By using QISM the eigenstates of the systems are constructed to derive the BA equations, which are the same as that derived by Sutherland (1968) only for fermions. The matrix periodic boundary conditions (PBC) for the fermions and bosons are obtained and the relationship between them is discussed. They are useful for the further study of the QNSM with $q\sim(m, n)\times(k, 0)$. Finally, the BA equations are generalised to the QNSM with the two-body mirror-image delta potentials, which was proposed by Schulz (1987) to solve a Kondo model with quadratic band energy.

2. Algebraic BA

The auxiliary linear equation in QISM connected with the model (1.1) is

$$\begin{aligned}&\frac{\partial}{\partial x}T(x, y\,|\,\lambda)=:L(x\,|\,\lambda)T(x, y\,|\,\lambda):\\&T(x, x\,|\,\lambda)=1\end{aligned}\tag{2.1}$$

where the colons : : mean normal product. λ is the spectral parameter and

$$L(x\,|\,\lambda)=\begin{pmatrix}\mathrm{i}\frac{1}{2}\lambda & \mathrm{i}\sqrt{c}q(x)\\ -\mathrm{i}\sqrt{c}q^{\dagger}(x) & -\mathrm{i}\frac{1}{2}\lambda\end{pmatrix}.\tag{2.2}$$

From the structure of q we know that both $L(x\,|\,\lambda)$ and $T(x, y\,|\,\lambda)$ have the supermatrix structure $(n, 1)\times(n, 1)$ or $(1, n)\times(1, n)$ and $(0, n+1)\times(0, n+1)$ or $(n+1, 0)\times(n+1, 0)$ for fermions and bosons respectively.

The monodromy matrix is

$$T_L(\lambda)=T(L, -L\,|\,\lambda)=\begin{pmatrix}A(\lambda)_{ab} & B(\lambda)_a\\ C(\lambda)_b & D(\lambda)\end{pmatrix}$$

where a and b may be taken as $1, 2, \ldots, n$ by our convention.

Using the auxiliary linear equation (2.1), we can find the following Yang-Baxter relation (YBR):

$$R(\lambda-\mu)T_L(\lambda)\underset{s}{\otimes}T_L(\mu)=T_L(\mu)\underset{s}{\otimes}T_L(\lambda)R(\lambda-\mu)\tag{2.3}$$

and

$$R(\lambda)=b(\lambda)+a(\lambda)P_{n+1}\qquad b(\lambda)=1-a(\lambda)=\frac{\mathrm{i}\kappa c}{\lambda+\mathrm{i}\kappa c}\tag{2.4}$$

where $\kappa=1$ for $q\sim(n, 0)\times(0, 1)$ and $(0, n)\times(0, 1)$, $\kappa=-1$ for $q\sim(0, n)\times(1, 0)$ and $(n, 0)\times(1, 0)$. P_{n+1} is the graded permutation operator

$$P_{n+1}=(-1)^{p(j)}E_{ij}\underset{s}{\otimes}E_{ji}.$$

The double indices i and j mean summations over $1, 2, \ldots, n+1$. $\otimes$ indicates the direct product and the symbol s below $\otimes$ indicates that the direct product is defined as that in Grassmann algebra. E_{ij} is an $(n+1)\times(n+1)$ matrix, for which only a matrix element (i, j) is unity and the others are zero.

YBR (2.3) enables us to give the commutation relation among $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$. We write out some of them, which play a fundamental role for the construction of the eigenstates of the systems. They are

$$A(\lambda)_{ab}C(\mu)_c = C(\mu)_{c'}A(\lambda)_{ab'}\frac{r(\lambda-\mu)_{c'b',bc}}{a(\lambda-\mu)} + \kappa p\frac{b(\lambda-\mu)}{a(\lambda-\mu)}C(\lambda)_b A(\mu)_{ac} \tag{2.5a}$$

$$D(\lambda)C(\mu)_c = C(\mu)_c D(\lambda)\left(1-\kappa\frac{b(\mu-\lambda)}{a(\mu-\lambda)}\right) - \kappa\frac{b(\lambda-\mu)}{a(\lambda-\mu)}C(\lambda)_c D(\lambda) \tag{2.5b}$$

$$C(\lambda)_a C(\mu)_b[b(\lambda-\mu)-a(\lambda-\mu)\kappa](-\kappa p) = C(\mu)_{b'}C(\lambda)_{a'}r(\lambda-\mu)_{b'a',ab} \tag{2.5c}$$

$$[\text{Str } T_L(\lambda), \text{Str } T_L(\mu)] = 0 \qquad \text{Str } T_L(\lambda) = -\kappa p \text{ tr } A(\lambda) - D(\lambda) \tag{2.6}$$

$$r(\lambda) = -\kappa p b(\lambda) + a(\lambda)P_n \tag{2.7}$$

where the double indices a', b', c' and a mean summations over $1, 2, \ldots, n$ by our convention. The permutation operator P_n is

$$P_n = e_{ab} \otimes e_{ba}. \tag{2.8}$$

e_{ab} is an $n \times n$ matrix in which element (a, b) equals unity and the others are zero.

Commutation relation (2.6) enables us to consider Str $T_L(\lambda)$ as a generating function of the infinite set of the conserved quantities. Using (2.1), we can expand the monodromy matrix $T_L(\lambda)$ as the Neumann series and further obtain the following asymptotic expansion (Pu and Zhao 1986):

$$\begin{aligned}
&\lim_{i\lambda\to-\infty} \ln[-\kappa \text{Str } T_L(\lambda)\exp(i\lambda L)] = C_1/i\lambda + C_2/(i\lambda)^2 + C^3/(i\lambda)^3 + \ldots \\
&C_1 = -cN \\
&C_2 = -icP - c^2N/2 \\
&C_3 = cH - ic^2P - c^3N/3
\end{aligned} \tag{2.9}$$

where all C_i, $i = 1, 2, \ldots$, are the conserved quantities of the system. Hence the Hamiltonian of the system is $H = C_3/c - C_2 + cC_1/6$. N and P are the particle number and momentum of the system respectively and have the form

$$N = \int \mathrm{d}x\, q^+ q \tag{2.10a}$$

$$P = -i\int \mathrm{d}x\, q^+ \,\mathrm{d}q/\mathrm{d}x. \tag{2.10b}$$

In order that the model be soluble with the aid of the algebriac BA, it is necessary that we find the eigenstates for the infinite set of the conserved quantities, or Str $T_L(\lambda)$. Define the vacuum state $|0\rangle$ by $q(x)_a|0\rangle = 0$, $a = 1, 2, \ldots, n$. It leads to

$$\begin{aligned}
&A(\lambda)|0\rangle = \exp(i\lambda L)|0\rangle \\
&D(\lambda)|0\rangle = \exp(-i\lambda L)|0\rangle \\
&B(\lambda)|0\rangle = 0 \\
&C(\lambda)|0\rangle \neq 0.
\end{aligned} \tag{2.11}$$

These could be obtained from the Neumann series of $T_L(\lambda)$ (Faddeev 1981, Thacker 1981). According to the commutation relations (2.5*a*) and (2.5*b*), we should consider the following state vectors:

$$|\lambda_1, \lambda_2, \ldots, \lambda_N, f\rangle = C(\lambda_1)_{a_1} C(\lambda_2)_{a_2} \ldots C(\lambda_N)_{a_N} |0\rangle f^{a_1 a_2 \ldots a_N} Y(\lambda_1 \ldots \lambda_N) \tag{2.12a}$$

$$Y(\lambda_1 \ldots \lambda_N) = \begin{cases} 1 & \text{for bosons} \\ \prod_{i<j=1}^{N} a(\lambda_j - \lambda_i)^{-1} & \text{for fermions} \end{cases}$$

and the state vectors satisfy

$$|\lambda_1 \ldots \lambda_i \ldots \lambda_j \ldots \lambda_N, f\rangle = p|\lambda_1 \ldots \lambda_j \ldots \lambda_i \ldots \lambda_N, f\rangle. \tag{2.12b}$$

In order that those are the eigenstates for Str $T_L(\lambda)$ it is necessary and sufficient that f be an eigenstate for the operator $t(\lambda)$

$$t(\lambda) f = \nu(\lambda) f \tag{2.13}$$

and that the momenta λ satisfy the equations

$$\exp(2i\lambda_j L) = \nu(\lambda_j)^{-1} \prod_{\substack{i=1 \\ \neq j}}^{N} \frac{\lambda_j - \lambda_i + ic}{\lambda_j - \lambda_i - ipc} \qquad j = 1, 2, \ldots, N. \tag{2.14}$$

The operators which appear in equation (2.13) are

$$t(\lambda) = \mathrm{Tr}\ T(\lambda) \tag{2.15a}$$

$$T(\lambda) = L(\lambda - \lambda_N) \ldots L(\lambda - \lambda_1) \tag{2.15b}$$

$$L(\lambda - \lambda_i) = \beta(\lambda - \lambda_i) P_n^i + \alpha(\lambda - \lambda_i) \tag{2.15c}$$

$$\beta(\lambda - \lambda_i) = 1 - \alpha(\lambda - \lambda_i) = \frac{-ipc}{\lambda - \lambda_i - ipc} \tag{2.16}$$

where the permutation operator P_n^i is considered as a $n \times n$ matrix

$$P_n^i = \begin{pmatrix} e_{11}^i & e_{21}^i & \ldots & e_{n1}^i \\ \vdots & & & \vdots \\ e_{1n}^i & e_{2n}^i & \ldots & e_{nn}^i \end{pmatrix}.$$

So $L(\lambda - \lambda_i)$ and $T(\lambda)$ are in the $n \times n$ matrices. Using the formula $P_n^i P_n^j = P_n^j P_{ij}$, we may rewrite the operator $t(\lambda)$ as

$$\begin{aligned} t(\lambda) &= X_{j+1,j} X_{j+2,j} \ldots X_{N,j} X_{1,j} \ldots X_{j-1,j} \\ X_{i,j} &= \alpha(\lambda_j - \lambda_i) + \beta(\lambda_j - \lambda_i) P_{ij} \end{aligned} \tag{2.15d}$$

where the permutation operator $P_{ij} = e_{ab}^i \otimes e_{ba}^j$.

In fact, the equations (2.13) and (2.14) represent the periodic boundary condition (PBC) of the system. The permutation operator P_{ij} acting on f generally is a $N! \times N!$ representation of the permutation group S_N. This may be reduced to a sum of irreducible representations by considering f of various symmetry types. Here we consider the system having n species of fermions or bosons. Hence the appropriate symmetry for (2.13) is $R = [N - M_1, \ldots, M_{n-2} - M_{n-1}, M_{n-1}]$.

The $\nu(\lambda_j)$ are functions of the λ, p and the representation R. Let R^* be the conjugate representation of R. From (2.15d) we have

$$\nu(\lambda_j, p, R) = \prod_{i \neq j} \frac{\lambda_j - \lambda_i + \mathrm{i}pc}{\lambda_j - \lambda_i - \mathrm{i}pc} \nu(\lambda_j, -p, R^*). \tag{2.17}$$

If R represents a boson system, then R^* describes a fermion system with the same species content. Hence the matrix PBC for fermions can be obtained from those for bosons by the transformation (2.17) and vice versa. Moreover, this is useful for the further study of the QNSM with $q \sim (m, n) \times (k, 0)$. The $\nu(\lambda_j)$ are independent of the factor κ. It shows clearly that the results are the same for the fermions $q \sim (n, 0) \times (0, 1)$ and $(0, n) \times (1, 0)$, or for the bosons $q \sim (0, n) \times (0, 1)$ and $(n, 0) \times (1, 0)$.

We can solve equation (2.13) by using the QISM (Kulish and Reshetikhin 1981), which is straightforward but rather lengthy. We write only the final results as the following:

$$\nu(\lambda) = \prod_i^N \alpha(\lambda - \lambda_i) \prod_i^{M_1} \alpha(\lambda - \lambda_i^{(1)})^{-1} \nu(\lambda)_{n-1} + \prod_i^{M_1} \alpha(\lambda_i^{(1)} - \lambda)^{-1} \tag{2.18a}$$

and $\nu(\lambda)_{n-1}$ is given by the recurrence relations

$$\nu(\lambda)_{n-m} = \prod_i^{M_m} \alpha(\lambda - \lambda_i^{(m)}) \prod_i^{M_{m+1}} \alpha(\lambda - \lambda_i^{(m+1)})^{-1} \nu(\lambda)_{n-m-1} + \prod_i^{M_{m+1}} \alpha(\lambda_i^{(m+1)} - \lambda)^{-1}$$

$$m = 1, 2, \ldots, n-2 \tag{2.18b}$$

$$\nu(\lambda)_1 = 1$$

and these $\lambda_i^{(m)}$ satisfy the following equations:

$$\prod_i^N \alpha(\lambda_j^{(1)} - \lambda_i) = -\prod_i^{M_1} \frac{\alpha(\lambda_j^{(1)} - \lambda_i^{(1)})}{\alpha(\lambda_i^{(1)} - \lambda_j^{(1)})} \prod_i^{M_2} \alpha(\lambda_i^{(2)} - \lambda_j^{(1)}) \qquad j = 1, 2, \ldots, M_1 \tag{2.19a}$$

$$\prod_i^{M_{m-1}} \alpha(\lambda_j^{(m)} - \lambda_i^{(m-1)}) = -\prod_i^{M_m} \frac{\alpha(\lambda_j^{(m)} - \lambda_i^{(m)})}{\alpha(\lambda_i^{(m)} - \lambda_j^{(m)})} \prod_i^{M_{m+1}} \alpha(\lambda_i^{(m+1)} - \lambda_j^{(m)})$$

$$j = 1, 2, \ldots, M_m \qquad m = 2, 3, \ldots, n-2 \tag{2.19b}$$

$$\prod_i^{M_{n-2}} \alpha(\lambda_j^{(n-1)} - \lambda_i^{(n-2)}) = -\prod_i^{M_{n-1}} \frac{\alpha(\lambda_j^{(n-1)} - \lambda_i^{(n-1)})}{\alpha(\lambda_i^{(n-1)} - \lambda_j^{(n-1)})} \qquad j = 1, 2, \ldots, M_{n-1}. \tag{2.19c}$$

Moreover we can find

$$r(\lambda_j - \lambda_{j+1})_{a_j a_{j+1}, a_j' a_{j+1}'} f^{a_1 \ldots a_j' a_{j+1}' \ldots a_N}$$

$$= (-\kappa p b(\lambda_j - \lambda_{j+1}) + a(\lambda_j - \lambda_{j+1})) f^{a_1 \ldots a_j a_{j+1} \ldots a_N}. \tag{2.20}$$

Using (2.5c) and (2.20), we can directly prove (2.12b).

Equations (2.19a)-(2.19c) and (2.14) or

$$\exp(2\mathrm{i}\lambda_j L) = -p \prod_i^N \frac{\lambda_j - \lambda_i + \mathrm{i}c}{\lambda_j - \lambda_i - \mathrm{i}pc} \prod_i^{M_1} \alpha(\lambda_i^{(1)} - \lambda_j) \qquad j = 1, 2, \ldots, N \tag{2.21}$$

are the BA equations for the fermions $p = -1$ and the bosons $p = 1$. $\alpha(\lambda)$ is given by (2.16). Substituting $\lambda_j^{(m)} = \Lambda_j^{(m)} - \frac{1}{2}\mathrm{i}mc$ into the BA equations, for the fermions $p = -1$ these will coincide with the results given by Sutherland (1968). For the bosons $p = 1$ our results as $n = 1$ will be reduced to those in the works given by Lieb and Liniger (1963), Faddeev (1981) and Thacker (1981) and as $n = 2$ to those given by Kulish (1981).

3. BA equations for the QNSM with $\delta(x+y)$ potentials

As a special case, we set q as

$$q(x)=\begin{bmatrix}\psi(x)\\ \psi(-x)\end{bmatrix} \qquad \psi(x)=\begin{bmatrix}\psi_1(x)\\ \psi_2(x)\end{bmatrix} \tag{3.1}$$

with the commutation relations

$$\psi_i(x)\psi_j^{\dagger}(y)+\psi_j^{\dagger}(y)\psi_i(x)=\delta_{ij}\delta(x-y)$$
$$\psi_i(x)\psi_j(y)+\psi_j(y)\psi_i(x)=0.$$

Substituting (3.1) into the model (1.1), we will get the second quantised form of the following model:

$$H_s=-\sum_{i=1}^{N} \mathrm{d}^2/\mathrm{d}x_i^2+2c\sum_{i\neq j}^{N}(\delta(x_i-x_j)+\delta(x_i+x_j)) \tag{3.2}$$

which was proposed by Schulz to discuss a Kondo model with quadratic band energy.

In this section, we will show that the BA equations (2.21) and (2.19a)-(2.19c) for $n=4$ and $p=-1$ can be generalised to the model (3.2). The monodromy matrix $T_L(\lambda)$ is defined as

$$T_L(\lambda)=T(L+\varepsilon,-L|\lambda)$$

where ε is an infinitesimal positive quantity. The same asymptotic expansion as (2.9) in the limit $\mathrm{i}\lambda\to-\infty$ for Str $T_L(\lambda)\exp(\mathrm{i}\lambda L)$ could be obtained. Moreover the asymptotic expansion of Str $T_L(\lambda)\exp(-\mathrm{i}\lambda L)$ in the limit $\mathrm{i}\lambda\to\infty$ gives the other conserved quantities. Notably, $\mathrm{tr}(\int \mathrm{d}x :qq^{\dagger}:\int \mathrm{d}x :q\mathrm{i}\,\mathrm{d}q^{\dagger}/\mathrm{d}x:)$ is one of them. Its eigenvalue can be obtained from the eigenvalue of Str $T_L(\lambda)\exp(-\mathrm{i}\lambda L)$ in the limit $\mathrm{i}\lambda\to\infty$ and it is

$$\sum_{m=1}^{3} a_m(M_1,M_2,M_3)\sum_{i=1}^{2M_m}\Lambda_i^{(m)}.$$

In fact this conserved quantity is identical to zero, and it leads to

$$\sum_{i=1}^{2M_m}\Lambda_i^{(m)}=0.$$

The momentum, particle number and Hamiltonian are respectively

$$P=-\mathrm{i}\int \mathrm{d}x\, q^{\dagger}\,\mathrm{d}q/\mathrm{d}x=0 \qquad N_s=\tfrac{1}{2}N=\tfrac{1}{2}\int \mathrm{d}x\, q^{\dagger}q$$

and

$$H_s=\tfrac{1}{2}H=\tfrac{1}{2}\int \mathrm{d}x[(\mathrm{d}q^{\dagger}/\mathrm{d}x)\,(\mathrm{d}q/\mathrm{d}x)+c:q^{\dagger}qq^{\dagger}q:]. \tag{3.3}$$

The monodromy matrix $T_L(\lambda)$ is the solution of the auxiliary linear equation (2.1) for $n=4$ and satisfies the YBR (2.3). Hence the BA equations of the model (3.2) or (3.3) could be directly obtained from the equations (2.21) and (2.19a)-(2.19c) for $n=4$ and $p=-1$ provided that

$$\lambda_j=k_j \qquad \lambda_{N_s+j}=-k_j \qquad j=1,2,\ldots,N_s$$

and

$$\lambda_j^{(m)} = \Lambda_j^{(m)} - \tfrac{1}{2}\mathrm{i}mc \qquad \lambda_{M_m+j}^{(m)} = -\Lambda_j^{(m)} - \tfrac{1}{2}\mathrm{i}mc$$

$$j = 1, 2, \ldots, M_m \qquad m = 1, 2, 3.$$

The particle number and the energy are N_s and $\Sigma_{i=1}^{N_s} k_i^2$ respectively.

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